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On Volterra Equations Driven by Semimartingales

CONSTANTIN TUDOR

Faculty of Mathematics, University of Bucharest, Str. Academiei 14, 70109 Bucharest, Romania

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1. INTRODUCTION

We consider here stochastic equations of the form

$$X_t = H_t + \int_0^t F(t, s, X)\, dZ_s, \tag{1.1}$$

where Z is a finite dimensional semimartingale, H is a càdlàg process, and F is some functional.

The particular case when F is independent of t (in this case (1.1) is known as the Doleans–Dade and Protter equation) has been considered by several authors (e.g., [3, 6, 8, 9, 10, 19, 22, 23, 27, 28, 29] for strong solutions and [9, 12, 20] for weak solutions). Also, for the Doleans–Dade and Protter equation the continuous dependence and the convergence of finite differences have been studied in [8, 9, 12, 19, 24]. The case of Volterra equations is considered in [1, 2, 11, 13, 14, 16, 25] (strong solutions) and in [13, 15] (weak solutions).

Based on the transformation rule established by Protter [25], a Gronwall type inequality, and a domination property for semimartingales, we extend here Protter's result [25] about the existence and pathwise uniqueness of strong solutions of (1.1) (Theorem 3.4). We give a unified formulation of Theorems 1 and 2 from [28] and use it in order to prove the existence and pathwise uniqueness of strong solutions for systems of stochastic equations with Volterra components (Theorems 3.7 and 3.8).

In Section 4 the convergence with respect to the compact convergence in probability of finite differences to the solution of (1.1) is proved (Theorem 4.1).

2. PRELIMINARIES

We will recall in this section the basic notations and definitions and some results that we will need.

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$(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t\geqslant 0})$ is a filtered probability space with the usual assumptions. For a process $(X_t)_{t\geqslant 0}$ and a $\mathscr{F}_t$-stopping time τ we define the processes X^τ (resp. $X^{\tau-}$) by

$$X_t^\tau = X_{t\wedge\tau};\ X_t^{\tau-} = X_{t1[0,\tau)} + X_{\tau-1[\tau,\infty)} \qquad (X_{0-}=0,\ X^{\tau-}=0 \text{ on } \tau=0).$$

The convergence is probability uniformly on compact intervals, defined on the space of càdlàg adapted processes, is denoted by $\mapsto^{cp}$. This convergence is given by the complete metric

$$\|X\| = \sum_{n\geqslant 1} 2^{-n} E[(\sup_{t\leqslant n} |X_t|)\wedge 1].$$

$D(R_+, R^d)$ is the class of càdlàg functions from R_+ to R^d endowed with the topology τ_u of uniform convergence on compact sets:

$$\mathscr{D}_t = \mathscr{B}(\{f; f\in D(R_+, R^d), f(s)\in B\}, s\leqslant t, B\in\mathscr{B}(R^d));$$

$$\mathscr{D} = \mathscr{B}\left(\bigcup_t \mathscr{D}_t\right); \mathscr{D}_{t+} = \bigcap_{s>t} \mathscr{D}_s;$$

$$\bar{\Omega} = \Omega\times D(R_+, R^d); \bar{\mathscr{F}} = \mathscr{F}\otimes\mathscr{D}; \bar{\mathscr{F}}_t = \mathscr{F}_t\otimes\mathscr{D}_{t+}.$$

Allowing Volterra integrands in (1.1) leads one naturally to consider an extension of the filtration $(\mathscr{F}_t)$ as follows.

Let $H(x, s, \omega)$: $R_+^2\times\Omega\mapsto R^d\otimes R^m$ be an application and Z be a R^m-valued semimartingale. Define

$$\mathscr{G}_t(H) = \bigcap_{s>t} (\mathscr{F}_s \wedge \mathscr{B}(H_{ij}(x, u, \cdot); 0\leqslant u\leqslant s, u\leqslant x, 1\leqslant i\leqslant d, 1\leqslant j\leqslant m))$$

$$\begin{aligned} A(Z) = \{H\colon R_+^2\times\Omega\mapsto R^d\otimes R^m;\ & E \text{ is a } \mathscr{G}(H)\text{-semimartingale},\\ & \text{for each } x,\ (x, s, \omega)\mapsto H(x, s, \omega) \text{ is } \mathscr{B}(R_+^2)\otimes\mathscr{G}(H)\text{-measurable}\\ & \text{and for every } s\leqslant x,\ i,\ (s,\omega)\mapsto H_i(x, s, \omega) \text{ is}\\ & \mathscr{G}(H)\text{-predictable and } Z\text{-integrable}\}. \end{aligned}$$

The following result is due to Protter [25, Theorem 3.3].

THEOREM 2.1 (Transformation Rule). *Let $H\in A(Z)$ be such that $\partial H(t, s, \omega)/\partial t$ exists, and is locally bounded uniformly in t. Then the process $Y_t = \int_0^t H(t, s)\, dZ_s$ is $\mathscr{G}(H)$-semimartingale and if $Z = M + A$ is a $\mathscr{G}(H)$-decomposition of Z then*

$$Y_t = \int_0^t H(s, s)\, dM_s + \left\{\int_0^t H(s, s)\, dA_s + \int_0^t \left(\int_0^s \partial H(s, u)/\partial s\, dZ_u\right) ds\right\}$$

is a decomposition of Y.

3. Existence and Uniqueness of Strong Solutions

Consider the following class of functions

$$LS = \left\{ \rho;\ \rho \text{ is strictly increasing, concave, and } \int_{0+}^{1} du/\rho(u) = \infty \right\}.$$

For example functions of the form $x\,|\log x|^{1-\varepsilon}$ in the neighborhood of 0 which are strictly increasing and concave are in LS. Observe that if ρ_1, ρ_2 are in LS and $\alpha, \beta \geqslant 0$, $\alpha + \beta > 0$ then $\alpha\rho_1 + \beta\rho_2 \in LS$. The following lemma of Gronwall type often arises in problems concerning the existence, uniqueness, stability, and approximation of solutions of stochastic differential equations.

Lemma 3.1. *Suppose we are given*

$\{A(t)\}_{t \geqslant 0}$ *an increasing adapted càdlàg process with* $A(t) \geqslant t$ *a.s. for each* $t \geqslant 0$,

$\{I_n(t)\}_{t\geqslant 0}$, $\{J_{m,n}(t)\}_{t\geqslant 0}$, $m, n = 1, 2, \ldots$, *sequences of nonnegative adapted càdlàg processes,*

$\rho, \rho_1, \rho_2 \in LS$ *and* (a_n) *a sequence of nonnegative numbers such that* $\overline{\lim}_n a_n = a < \infty$.

Define the stopping times $\theta(t) = \inf(s; A(s) > t)$, $\delta(t) = \inf(s; A(s) \geqslant t)$ (*note that* $\delta(t) \leqslant \theta(t) \leqslant t$, $\delta(t) \nearrow \infty$ *as* $t \to \infty$).

(i_1) *Assume that for any* n, T *there is a constant* $C(T)$ *such that for every stopping time* $\theta \leqslant \theta(T)$

$$E[\sup_{t < \theta(T)} I_n(t)] < \infty \tag{3.1}$$

$$E[I_n(\theta-)] \leqslant a_n + C(T) E\left[\int_{[0,\theta)} \rho(I_n(t-))\, dA(t)\right]. \tag{3.2}$$

Then for some constant $C(a, T, \rho)$ *depending only on* a, T, ρ (*with* $C(0, T, \rho) = 0$)

$$\overline{\lim_n} \sup_{t \leqslant T} E[I_n(\theta(t)-)] \leqslant C(a, T, \rho) < \infty. \tag{3.3}$$

Moreover, if every I_n *is increasing and* $a = 0$, *then* $I_n \mapsto^{cp} 0$ *as* $n \to \infty$.

(i_2) *Assume that for every* m, n, T, *and stopping time* $\theta \leqslant \theta(T)$

$$\sup_{r,s} E[\sup_{t < \theta(T)} J_{r,s}(t)] < \infty \tag{3.4}$$

$$E[J_{m,n}(\theta-)] \leqslant C(T) E\left\{\int_{[0,\theta)} [\rho_1(J_{m,n}(t-)) + \rho_2(J_{m-1,n-1}(t-))]\, dA(t)\right\}. \tag{3.5}$$

Then

$$\lim_{m,n} \sup_{t \leqslant T} E[J_{m,n}(\theta(t)-)] = 0.$$

In particular if every $J_{m,n}$ *is increasing then* $J_{m,n} \mapsto^{cp} 0$ *as* $m, n \mapsto \infty$.

(j_1) *Let* A *be predictable and* $a = 0$. *Asssume that for every* n, T, *and stopping times* $\sigma \leqslant \delta$, *with* δ *predictable and* $\delta \leqslant \delta(T)$,

$$E[\sup_{t < \delta(T)} I_n(t)] < \infty \tag{3.6}$$

$$E[I_n^\sigma(\delta-)] \leqslant a_n + C(T) E\left[\int_{[0,\theta)} \rho(I_n^\sigma(t-))\, dA(t)\right]. \tag{3.7}$$

Then $I_n \mapsto^{cp} 0$ *as* $n \mapsto \infty$.

(j_2) *Let* A *be predictable. Assume that for every* m, n, T, *and stopping times* $\sigma \leqslant \delta$, *with* δ *predictable and* $\delta \leqslant \delta(T)$,

$$\sup_{r,s} E[\sup_{t < \delta(T)} J_{r,s}(t)] < \infty \tag{3.8}$$

$$E[J_{m,n}^\sigma(\delta-)] \leqslant C(T) E\left\{\int_{[0,\delta)} [\rho_1(J_{m,n}(t-)) + \rho_2(J_{m-1,n-1}(t-))]\, dA(t)\right\} \tag{3.9}$$

Then $J_{m,n} \mapsto^{cp} 0$ *as* $m, n \mapsto \infty$.

Proof. Utilising the time change theorem, Jensen's inequality, and $A(\theta(s)-) \leqslant s$ we obtain for $t \leqslant T$

$$E[I_n(\theta(t)-)] \leqslant a_n + C(T) \int_0^t \rho(E[I_n(\theta(s)-)])\, ds. \tag{3.10}$$

Let $\alpha, \beta > 0$ be such that $\rho(x) \leqslant \alpha + \beta x$. Then from (3.10) we have

$$E[I_n(\theta(t)-)] \leqslant a_n + K(T)\left[1 + \int_0^t E(I_n(\theta(s)-))\, ds\right]$$

so that

$$E[I_n(\theta(t)-)] \leqslant (a_n + K(T)) \exp\{K(T)T\},$$

which from (3.3) is immediate if $a \neq 0$.

Now assume $a = 0$. From (3.10) it follows

$$\overline{\lim_n} \sup_{s \leqslant t} E[I_n(\theta(s)-)]$$

$$\leqslant C(T) \int_0^t \rho(\overline{\lim_n} \sup_{u \leqslant s} E[I_n(\theta(u)-)])\, ds$$

and this implies that

$$\overline{\lim_{n}} \sup_{s \leqslant t} E[I_n(\theta(s)-)] = 0.$$

The rest of the proof follows easily.

(i_2) From (3.5) we have for $t \leqslant T$

$$\overline{\lim_{m,n}} \sup_{s \leqslant t} E[J_{m,n}(\theta(s)-)]$$
$$\leqslant K(T) \int_0^t (\rho_1 + \rho_2)(\overline{\lim_{m,n}} \sup_{u \leqslant s} E[J_{m,n}(\theta(u)-)]\, ds$$

so that

$$\overline{\lim_{m,n}} \sup_{s \leqslant t} E[J_{m,n}(\theta(s)-)] = 0.$$

(j_1) Choose a sequence of stopping times $\alpha(k) < \delta(k)$, $\alpha(k) \nearrow \infty$ and define $\sigma(n) = \inf(t; I_n(t) \geqslant \varepsilon)$. Observe that, for every $h: R_+ \mapsto R_+$ measurable, the following inequality holds

$$\int_{[0,\delta(t))} h(s)\, dA(s) \leqslant \int_0^t h(\delta(s))\, ds. \tag{3.11}$$

The computation made in (i_1) (with $\theta(s)$ replaced by $\delta(s)$) and (3.11) imply

$$\overline{\lim_{n}}\, E[I_n^{\sigma(n) \wedge \sigma}(\delta(t)-)] = 0$$

for all $t \geqslant 0$ and stopping time σ, where from taking $\sigma = \alpha(k)$ we get

$$\overline{\lim_{n}}\, E[I_n(\sigma(n) \wedge \alpha(k))] = 0 \qquad \text{for every } k.$$

Then

$$P(\sup_{t \leqslant \alpha(k)} I_n(t) \geqslant \varepsilon) \leqslant P(I_n(\sigma(n) \wedge \alpha(k)) \geqslant \varepsilon)$$
$$\leqslant \varepsilon^{-1} E[I_n(\sigma(n) \wedge \alpha(k))] \mapsto 0 \text{ as } n \mapsto \infty, \text{ for every } k.$$

Therefore $I_n \mapsto^{cp} 0$ as $n \mapsto \infty$.

(j_2) Similarly as in (i_2).

Remark 3.2. If A is predictable and (3.1), (3.2) hold only for every predictable stopping time $\theta \leqslant \delta(T)$, then (3.3) holds with $\theta(t)$ replaced by $\delta(t)$.

LEMMA 3.3 (Domination Propery). *Let H, Z be as in Theorem 2.1 and $2 \leqslant p < \infty$. Then there exists an increasing process L(L is continuous if Z is so) with $L_t \geqslant t$ and controlling Z in the following sense: for any $\mathscr{G}(H)$-stopping times $\sigma \leqslant \tau$ bounded a.s. by a constant T*

$$\begin{aligned} E(\sup_{\sigma \leqslant t < \tau} |Y_t - Y_\sigma|^p) &\leqslant C(p, T)\left\{E\left[\left(L_{\tau-}\int_{(\sigma,\tau)} |H(s,s)|^2\, dL_s\right)^{p/2}\right]\right. \\ &\quad + [E(|L_{\tau-} - L_\sigma|^{2(p-1)})]^{1/2} \\ &\quad \left.\times\left[\int_0^T E\left(L_{\tau-}\int_{(0,\tau)} |\partial H(s,x)/\partial s|^2\, dL_s\right)^p ds\right]^{1/2}\right\}. \end{aligned}$$

In particular if $|H(t, s, \omega)| + |\partial H(t, s, \omega)/\partial s| \leqslant K$ then

$$\begin{aligned} E(\sup_{\sigma \leqslant t < \tau} |Y_t - Y_\sigma|^p) \leqslant C(p, T, K)\{&E[L_{\tau-}^{p/2}(L_{\tau-} - L_\sigma)^{p/2}] \\ &+ [E(L_{\tau-}^{2p})]^{1/2}[E(|L_{\tau-} - L_\sigma|^{2(p-1)})]^{1/2}\}. \end{aligned}$$

Proof. By a result of Pratelli [21, Corollary 3] there is an increasing process Q(Q is continuous if Z is so) such that

$$\begin{aligned} &E\left[\sup_{\sigma \leqslant t < \tau}\left|\int_{(\sigma,t]} H(s,s)\, dZ_s\right|^p\right] \\ &\qquad \leqslant C(p)E\left[\left(Q_{\tau-}\int_{(\sigma,\tau)} |H(s,s)|^2\, dQ_s\right)^{p/2}\right]. \end{aligned}$$

Next by Hölder's, Schwartz's, and Pratelli's inequalities we have

$$\begin{aligned} &E\left[\sup_{\sigma \leqslant t < \tau}\left|\int_{(\sigma,t]}\left(\int_0^s \partial H(s,x)/\partial s\, dZ_x\right) ds\right|^p\right] \\ &\quad \leqslant E\left[(\tau - \sigma)^{p-1}\int_0^T \sup_{t<\tau}\left|\int_0^t \partial H(s,x)/\partial s\, dZ_x\right|^p ds\right] \\ &\quad \leqslant T[E(|\tau - \sigma|^{2(p-1)})]^{1/2} \\ &\qquad \times\left[\int_0^T E\left(\sup_{t<\tau}\left|\int_{(0,t]} \partial H(s,x)/\partial s\, dZ_x\right|^{2p} ds\right]^{1/2}\right. \\ &\quad \leqslant C(p,T)[E(|\tau - \sigma|^{2(p-1)})]^{1/2} \\ &\qquad \times\left\{\int_0^T E\left[\left(Q_{\tau-}\int_{(0,\tau)} |\partial H(s,x)/\partial s\, ds|^2\, dQ_x\right)^p ds\right]\right\}^{1/2}. \end{aligned}$$

The process $L_t = t + Q_t$ satisfies the requirements.

THEOREM 3.4. *Let $F: R_+^2 \times \bar{\Omega} \mapsto R^d \otimes R^m$ be such that*

(1) *F is $\mathscr{B}(R_+^2) \otimes \bar{\mathscr{F}}$-measurable;*

(2) *If $\mathscr{G}_s(F) = \bigcap_{u>s}[\mathscr{F}_u \vee \mathscr{B}(F(t, r, f);\ 0 \leqslant r \leqslant u,\ u \leqslant t,\ f \in D(R_+, R^d))]$ then the process $\{F(t, s, \cdot)\}_{s \leqslant t}$ is $\overline{\mathscr{G}(F)}$-predictable;*

(3) *There exists $\rho \in LS$ and $\mathscr{F}$-predictable nonnegative processes γ, γ^r locally L-integrable (L is given by Lemma 3.3) such that*

$$|F(s, s, f) - F(s, s, g)|^2 \leqslant \gamma^r(s)\, \rho(\sup_{u<s} |f(u) - g(u)|^2) \tag{3.12}$$

for any $r > 0$, $f, g \in D_r(R_+, R^d)$

$$|F(s, s, f)|^2 \leqslant \gamma(s)[1 + \sup_{u<s} |f(u)|^2] \qquad \text{for any} \quad f \in D(R_+, R^d); \tag{3.13}$$

(4) *$\partial F(t, s, f)/\partial t$ exists and there are $\rho_1 \in LS$, $\gamma_1: R_+ \mapsto R_+$ locally integrable with respect to the Lebesgue measure and the processes γ_2, γ_2^r which are nonnegative, $\mathscr{G}(F)$-predictable, locally L-integrable such that*

$$|\partial F(t, s, f)/\partial t - \partial F(t, s, g)/\partial t|^2 \leqslant \gamma_1(t)\, \gamma_2^r(s)\, \rho_1(\sup_{u<s} |f(u) - g(u)|^2) \tag{3.14}$$

for every $f, g \in D_r(R_+, R^d)$,

$$|\partial F(t, s, f)/\partial t|^2 \leqslant \gamma(t)\, \gamma_2(s)[1 + \sup_{u<s} |f(u)|^2] \tag{3.15}$$

for any $f \in D(R_+, R^d)$.

Then, for every càdlàg and $\mathscr{G}(F)$-adapted R^d-valued process H, there exists a pathwise unique solution of (1.1).

Proof. Existence. By a result of Dellacherie [5, Theorem 57, p. 246] we may assume that for every T

$$E(\sup_{t \leqslant T} |H_t|^2) < \infty$$

(eventually we replace the probability P by an equivalent one; X is a solution of (1.1) if and only if X is a solution under the new probability measure). Also by the standard method we may assume that $\gamma^r = \gamma$, $\gamma_2^r = \gamma_2$ for every r and that (3.12), (3.14) hold for every $f, g \in D(R_+, R^d)$. Define $\tilde{L}_t = \int_0^t [1 + \gamma(s) + \gamma_2(s)]\, dL_s$; $\theta(s) = \inf(t; \tilde{L}_t > s)$; and the successive approximations

$$X^0 = H, \qquad X_t^{n+1} = H_t + \int_0^t F(t, s, X^n)\, dZ_s. \tag{3.16}$$

Utilising the domination property and (3.12)–(3.15) we have for $s \leqslant T$

$$E(\sup_{t<\theta(s)} |X_t^m - X_t^n|^2)$$
$$\leqslant C(T)E\left[\int_{[0,\theta(s))} (\rho_1+\rho_2)(\sup_{u<t} |X_u^{m-1} - X_u^{n-1}|^2)\, d\tilde{L}_t\right] \quad (3.17)$$
$$E(\sup_{t<\theta(s)} |X_t^n|^2)$$
$$\leqslant C_1(T)\left\{1 + E\left[\int_{[0,\theta(s))} \sup_{u<t} |X_u^{n-1}|^2\, d\tilde{L}_u\right]\right\}$$
$$\leqslant C_2(T)\left[1 + \int_0^s E(\sup_{u<\theta(t)} |X_u^{n-1}|^2)\, dt\right]. \quad (3.18)$$

An induction argument implies that $E(\sup_{u<\theta(s)} |X_u^n|^2) < \infty$ for any n, s. An iteration of (3.18) yields

$$\sup_n \sup_{s\leqslant T} E(\sup_{t<\theta(s)} |X_t^n|^2) \leqslant C_3(T) < \infty.$$

Now from (3.17) and Lemma 3.1(i_1) we deduce that $X^m - X^n \mapsto^{cp} 0$ as $m, n \mapsto \infty$ and in particular $X^n \mapsto^{cp} X$.

The next lemma justifies that we can pass to the limit in (3.16) with respect to the compact convergence in probability and thus X follows a solution of (1.1).

LEMMA 3.5. *Suppose the hypotheses of Theorem 3.4 are satisfied. Then* $\int_0^\cdot F(\cdot, s, X^n)\, dZ_s \mapsto^{cp} \int_0^\cdot F(\cdot, s, X)\, dZ_s$.

Proof. By the dominated convergence theorem for stochastic integrals (see [5, Number 14, Remark 6, p. 323]) we have

$$\int_0^\cdot F(s, s, X^n)\, dZ_s \overset{cp}{\longmapsto} \int_0^\cdot F(s, s, X)\, dZ_s.$$

Let $\eta(k) \nearrow \infty$, $\eta(k)$ stopping times, such that for every k

$$E[\sup_{t<\eta(k)} |X_t^n - X_t|^2] \mapsto 0$$

(eventually we take a subsequence in n). Denoting $\tilde{\theta}(k) = \theta(k) \wedge \eta(k)$ we obtain for fixed k

$$E\left\{\sup_{t<\bar{\theta}(k)}\left|\int_0^t\left(\int_0^s [\partial F(s,u,X^n)/\partial s-\partial F(s,u,X)/\partial s]\,dZ_u\right)ds\right|^2\right\}$$
$$\leqslant C(k)\int_0^k E\left[\int_{[0,\bar{\theta}(k))}|\partial F(s,u,X^n)/\partial s-\partial F(s,u,X)/\partial s|^2\,dL_u\right]ds$$
$$\leqslant C_1(k)E\left[\int_{[0,\bar{\theta}(k))}\rho(\sup_{s<t}|X_s^n-X_s|^2)\,d\tilde{L}_t\right]$$
$$\leqslant C_2(k)\,\rho(E[\sup_{t<\eta(k)}|X_t^n-X_t|^2])\mapsto 0 \quad \text{as} \quad n\mapsto\infty.$$

Therefore the conclusion follows easily by the transformation rule.

Uniqueness. Let X_1, X_2 be two solutions. Without loss of generality we may assume that for every T

$$E[\sup_{t\leqslant T}|X_i(t)|^2]<\infty.$$

Utilising the domination property we deduce for every stopping time $\theta\leqslant\theta(T)$

$$E[\sup_{t<\theta}|X_1(t)-X_2(t)|^2]$$
$$\leqslant C(T)E\left[\int_{[0,\theta)}(\rho_1+\rho_2)(\sup_{s<t}|X_1(s)-X_2(s)|^2)\,d\tilde{L}_u\right],$$

where from we conclude by Lemma 3.1(i_1). The proof of the theorem is finished.

Remark 3.6. Theorem 3.4 extends the result of Protter [25, Theorem 4.3] and of McShane [17, Theorem 11.2] (the McShane differentials are quasi-left-continuous semimartingales as is shown in [26]). The following theorem represents and unified formulation of Theorems 1 and 2 from [28].

THEOREM 3.7. *Let Z_i, $i=1,2$, be R^{m_i}-valued $\mathscr{F}$-semimartingales and let H_i be R^{d_i}-valued càdlàg $\mathscr{F}$-adapted processes. Let $F_i\colon R_+\times\bar{\Omega}\mapsto R^{d_i}\otimes R^{m_i}$ be $\bar{\mathscr{F}}$-predictable functionals. Let (A, B, ν) be the local characteristics of the $R^{m_1+m_2}=R^m$-valued semimartingale $Z=(Z_1, Z_2)$ and define the $\mathscr{F}$-predictable increasing process*

$$V(t)=\sum_{k\leqslant m}\left(\int_0^t|dA_k(s)|+B_{kk}(t)\right)+\int_{R^m}\nu([0,t],dz)(|z|^2\wedge 1).$$

Assume that

(1) *Z_2 is special with the canonical decomposition $Z_2 = M_2 + A_2$; denote by $\alpha = (\alpha^{jk})$, $\beta = (\beta^j)$ the $\mathscr{F}$-predictable processes given by the factorisations $\langle M_2^j, M_2^k \rangle = \alpha^{jk} \cdot V$; $A_2 = \beta \cdot V$.*

(2) *For every positive integer n there exist $\rho^n \in LS$ and the nonnegative $\mathscr{F}$-predictable processes γ^n locally V-integrable such that for every $(t, \omega) \in R_+ \times \Omega$, $f = (f_1, f_2)$, $g = (g_1, g_2)$ from $D_n(R_+, R^d)$ (denoting by DF_2 the matrix $F_2(t, \omega, f) - F_2(t, \omega, g)$ and by DF_2' its transposition)*

$$\begin{aligned} &|F_1(t, \omega, f) - F_1(t, \omega, g)|^2 + 2\langle f_2(t-) - g_2(t-), DF_2 \beta(t, \omega) \rangle \\ &\quad + \Delta V(t, \omega)\, |DF_2 \beta(t, \omega)|^2 + DF_2 \alpha(t, \omega)\, DF_2' \\ &\quad \leqslant \gamma^n(t, \omega)\, \rho^n(\sup_{s<t} |f_1(s) - g_1(s)|^2 + |f_2(t-) - g_2(t-)|^2). \end{aligned} \tag{3.19}$$

(3) *There exists a nonnegative $\mathscr{F}$-predictable and locally V-integrable process γ such that for every $(t, \omega) \in R_+ \times \Omega$, $f \in D(R_+, R^d)$*

$$|F_1(t, \omega, f)|^2 + |F_2(t, \omega, f)|^2 \leqslant \gamma(t, \omega)[1 + \sup_{s<t} |f(s)|^2]. \tag{3.20}$$

(4) *For every $(t, \omega) \in R_+ \times \Omega$, $F_i(t, \omega, \cdot)$ are τ_u-continuous.*

Then there exists one and only one strong solution of

$$\begin{aligned} X_1(t) &= H_1(t) + \int_0^t F_1(s, X)\, dZ_1(s) \\ X_2(t) &= H_2(t) + \int_0^t F_2(s, X)\, dZ_2(s). \end{aligned} \tag{I}$$

Proof. Since we have guaranteed the existence of a very good solution of (I) under (1), (3), (4) (see [12, 20]) the proof will follow if we prove that every two solutions X, Y, defined on the same good extension, are indistinguishable. Define

$$V^n(t) = \int_0^t \gamma^n(s)\, dV(s) + t$$

$$\sigma(n) = \inf(t;\ |X(t)| \geqslant n \text{ or } |Y(t)| \geqslant n)$$

$$\delta(n, k) = \inf(t;\ V^n(t) \geqslant k)$$

$$\tilde{X} = X^{\sigma(n)-}; \qquad \tilde{Y} = Y^{\sigma(n)-}$$

$$I(t) = \sup_{s \leqslant t} |\tilde{X}_1(s) - \tilde{Y}_1(s)|^2 + |\tilde{X}_2(t) - \tilde{Y}_2(t)|^2.$$

Of course $\tilde{X}, \tilde{Y} \in D_n(R_+, R^d)$. By the domination property we deduce for every stopping time $\delta \leqslant \delta(n, k)$

$$E[\sup_{t<\delta} |\tilde{X}_1(t) - \tilde{Y}_1(t)|^2] \leqslant C(k) E\left[\int_{[0,\delta)} |F_1(t, \tilde{X}) - F_1(t, \tilde{Y})|^2 \, dV(t)\right]. \tag{3.21}$$

By Ito's formula we obtain

$$\begin{aligned} E[|\tilde{X}_2(\delta-) - \tilde{Y}_2(\delta-)|^2] &\leqslant E\left\{\int_{[0,\delta)} [2\langle \tilde{X}_2(t-) - \tilde{Y}_2(t-), DF_2(t, \tilde{X}, \tilde{Y})\, \beta(t)\rangle \right. \\ &\quad + \Delta V(t)\, |DF_2(t, \tilde{X}, \tilde{Y})\, \beta(t)|^2 \\ &\quad \left. + DF_2(t, \tilde{X}, \tilde{Y})\, \alpha(t)\, DF_2'(t, \tilde{X}, \tilde{Y})] \, dV(t)\right\}. \end{aligned} \tag{3.22}$$

Summing (3.21), (3.22) and utilising (3.19) we get

$$E[I(\delta-)] \leqslant C(k) E\left[\int_{[0,\delta)} \rho^n(I(t-)) \, dV^n(t)\right]$$

for every predictable stopping time $\delta \leqslant \delta(n, k)$.

By Lemma 3.1(i_1) (see also Remark 3.2) we obtain $I(\delta(n, k)-) = 0$ for every n, k, where from X and Y are indistinguishable.

THEOREM 3.8. *Let Z_i, $i = 1, 2$, be R^{m_i}-valued $\mathscr{F}$-semimartingales and $F_1 : R_+^2 \times \bar{\Omega} \mapsto R^{d_1} \otimes R^{m_1}$, $F_2 : R_+ \times \bar{\Omega} \mapsto R^{d_2} \otimes R^{m_2}$ be some functionals. Assume that*

(1) *F_1 is $\mathscr{B}(R_+^2) \otimes \bar{\mathscr{F}}$-measurable;*

(2) *$\{F_1(t, s, \omega, f)\}_{s \leqslant t}$ is $\overline{\mathscr{G}(F_1)}$-predictable and F_2 is $\bar{\mathscr{F}}$-predictable;*

(3) *$Z = (Z_1, Z_2)$ is $\mathscr{G}(F_1)$-semimartingale and Z_2 is special (define V, α, β as in Theorem 3.7, where Z is considered as $\mathscr{G}(F_1)$-semimartingale);*

(4) *There exist $\rho_1 \in LS$ and the $\mathscr{G}(F_1)$-predictable processes $\gamma_1, \tilde{\gamma}_1$ locally V-integrable such that for every $(t, \omega) \in R_+ \times \Omega$, $f, g \in D(R_+, R^d)$*

$$\begin{aligned} &|F_1(t, t, \omega, f) - F_1(t, t, \omega, g)|^2 \\ &\quad + 2\langle f_2(t-) - g_2(t-), DF_2(t, \omega, f, g)\, \beta(t, \omega)\rangle \\ &\quad + \Delta V(t, \omega)\, |DF_2(t, \omega, f, g)\, \beta(t, \omega)|^2 \\ &\quad + DF_2(t, \omega, f, g))\, \alpha(t, \omega)\, DF_2'(t, \omega, f, g) \\ &\leqslant \gamma_1(t, \omega)\, \rho_1(\sup_{s<t} |f_1(s) - g_1(s)|^2 + |f_2(t-) - g_2(t-)|^2) \end{aligned} \tag{3.23}$$

$$|F_1(t, t, \omega, f)|^2 + |F_2(t, \omega, f)|^2 \leqslant \tilde{\gamma}_1(t)[1 + \sup_{s<t} |f(s)|^2]; \tag{3.24}$$

(5) $\partial F_1(t, s, \omega, f)/\partial t$ *exists and there are* $\rho_2 \in LS$, γ_2, $\tilde{\gamma}_2 : R_+ \mapsto R_+$ *locally integrable with respect to the Lebesgue measure and* $\mathscr{G}(F_1)$-*predictable locally V-integrable processes* $\gamma_3, \tilde{\gamma}_3$ *such that for every* $(t, s, \omega) \in R_+^2 \times \Omega$, f, $g \in D(R_+, R^d)$

$$\begin{aligned} &|\partial F_1(t, s, \omega, f)/\partial t - \partial F_1(t, s, \omega, g)/\partial t|^2 \\ &\qquad \leqslant \gamma_2(t)\, \gamma_3(s)\, \rho_2(\sup_{u<s} |f_1(u) - g_1(u)|^2 + |f_2(s-) - g_2(s-)|^2) \end{aligned} \tag{3.25}$$

$$|\partial F_1(t, s, \omega, f)/\partial t|^2 \leqslant \tilde{\gamma}_2(t)\, \tilde{\gamma}_3(s, \omega)[1 + \sup_{s<t} |f(s)|^2]. \tag{3.26}$$

Then for all R^{d_i}-*valued processes* H_i *which are càdlàg and* $\mathscr{G}(F_1)$-*adapted there exists one and only one strong solution of*

$$\begin{aligned} X_1(t) &= H_1(t) + \int_0^t F_1(t, s, X)\, dZ_1(s) \\ X_2(t) &= H_2(t) + \int_0^t F_2(s, X)\, dZ_2(s). \end{aligned} \tag{II}$$

Proof. From Theorem 3.7 it follows that for every R^d-valued càdlàg and $\mathscr{G}(F_1)$-adapted process the system

$$\begin{aligned} X_1(t) &= H_1(t) + \int_0^t F_1(s, s, X)\, dZ_1(s) \\ &\quad + \int_0^t \left(\int_0^s \partial F_1(s, u, Y)/\partial s\, dZ_1(u) \right) ds \\ X_2(t) &= H_2(t) + \int_0^t F_2(s, X)\, dZ_2(s) \end{aligned}$$

has a pathwise unique strong solution.

Consider the successive approximations

$$\begin{aligned} X_0 &= (H_1, H_2); \\ X_1^{n+1}(t) &= H_1(t) + \int_0^t F_1(s, s, X^{n+1})\, dZ_1(s) \\ &\quad + \int_0^t \left(\int_0^s \partial F_1(s, u, X^n)/\partial s\, dZ_1(u) \right) ds \\ X_2^{n+1}(t) &= H_2(t) + \int_0^t F_2(s, X^{n+1})\, dZ_2(s). \end{aligned} \tag{3.27}$$

Define

$$I_{mn}(t) = \sup_{s \leqslant t} |X_1^m(s) - X_1^n(s)|^2 + |X_2^m(t) - X_2^n(t)|^2$$

$$\tilde{V}(t) = \int_0^t [\gamma_1(s) + \tilde{\gamma}_1(s) + \gamma_3(s) + \tilde{\gamma}_3(s) + \tilde{\gamma}_3(s)]\, dV(s) + t$$

$$\delta(t) = \inf(s;\ \tilde{V}(s) \geqslant t).$$

Without loss of generality we may assume that for every T

$$E[\sup_{t \leqslant T} |X_0(t)|^2] < \infty.$$

By the domination property we have for all stopping times $\delta \leqslant \delta(T)$, σ

$$E[\sup_{t<\delta} |X^{n+1}(t)|^2]$$
$$\leqslant C(T)\left\{1 + E\left[\int_{[0,\delta)} (\sup_{s<t} |X^{n+1}(s)|^2 + \sup_{s<t} |X^n(s)|^2)\, d\tilde{V}(t)\right]\right\} \tag{3.28}$$

$$E[I_{mn}^\sigma(\delta -)]$$
$$\leqslant C_1(T) E\left[\int_{[0,\delta)} (\rho_1(I_{mn}(t-)) + \rho_2(I_{m-1,n-1}(t-)))\, d\tilde{V}(t)\right]. \tag{3.29}$$

If we take in (3.28) $\delta = \delta(t)$ and we make an inductive argument we get

$$\sup_n E[\sup_{t<\delta(T)} |X^{n+1}(t)|^2] < \infty.$$

By Lemma 3.1(j_2) we conclude that $X^m - X^n \mapsto^{cp} 0$ as $m, n \mapsto \infty$. In particular $X^n \mapsto^{cp} X$ and by Lemma 3.5 it follows that X is a solution of (II). Uniqueness follows as in Theorem 3.4.

4. Convergence of Finite Differences

Let $\delta = \{t_0, t_1, ...\}$ be a partition of R_+ and $|\delta| = \max_k(t_{k+1} - t_k)$. Define the following approximating processes associated with (1.1)

$$X_1^\delta(0) = H(0); \quad X_1^\delta(t) = H(t) + \int_0^t F^\delta(t, s, X_1^\delta)\, dZ(s);$$

$$F^\delta(u, t, f) = F(u, t_k, f) \qquad \text{if} \quad t \in [t_k, t_{k+1})$$

$$X_2^\delta(0)=H(0); \qquad X_2^\delta(t_{k+1})=H(t_{k+1})+\int_0^{t_{k+1}} F(t_{k+1}, s, X_2^\delta)\,dZ(s)$$

$$X_2^\delta(t)=X_2^\delta(t_k) \qquad \text{if} \quad t\in[t_k, t_{k+1})$$

$$X_3^\delta(t)=H(t)+\int_0^t F(t, s, X_2^\delta)\,dZ(s).$$

THEOREM 4.1. *Suppose the assumptions of Theorem* 3.4 *are satisfied and let X be the strong solution of* (1.1).

(a) *If γ, γ_2 are increasing and there exist $\bar{\gamma}\colon R_+ \mapsto R_+$ locally integrable with respect to the Lebesgue measure and for every n an application $h^n\colon R_+ \mapsto R_+$ with $h^n(0)=0$ such that for every $s, t\leqslant u$, $f\in D_n(R_+, R^d)$*

$$|F(u,t,f)-F(u,s,f)|^2+|\partial F(u,t,f)/\partial u-\partial F(u,s,f)/\partial u|^2$$
$$\leqslant \bar{\gamma}(u)\, h^n(|t-s|).$$

Then $X_1^\delta \mapsto^{cp} X$ as $|\delta| \mapsto 0$.

(b) *If H, Z are continuous then $X_3^\delta \mapsto^{cp} X$ as $|\delta|\mapsto 0$.*

Proof. (a) Define $\alpha(k)=\inf(s; |H(s)|>k)$,

$$\tilde{L}_t=\int_0^t (1+\gamma(s)+\gamma_2(s))\,dL_t; \qquad \theta(t)=\inf(s; \tilde{L}_s>t).$$

Then utilising the hypotheses (3), (4) of Theorem 3.4 and the domination property we deduce for every stopping time $\theta\leqslant\theta(T)$

$$E[\sup_{t<\theta\wedge\alpha(k)} |X(t)|^2]\leqslant C(T,k)\left\{1+E\left[\int_{[0,\theta\wedge\alpha(k))} \sup_{s<t} |X(s)|^2\,d\tilde{L}_t\right]\right\},$$

where from by Lemma 3.1(i_1)

$$E[\sup_{t<\theta(T)\wedge\alpha(k)} |X(t)|^2]\leqslant C_1(T,k)<\infty. \tag{3.31}$$

Similarly we obtain

$$E[\sup_{t<\theta(T)\wedge\alpha(k)} |X_1^\delta(t)|^2]\leqslant C_2(T,k)<\infty. \tag{3.32}$$

Define $\sigma(r,\delta)=\inf(t; |X(t)|>r$ or $|X_1^\delta(t)|>r)$, $\sigma=\sigma(r,\delta)\wedge\alpha(k)$

$$\tilde{L}_t^r=\int_0^t [1+\gamma(s)+\gamma^r(s)+\gamma_2(s)+\gamma_2^r(s)]\,dL_s;$$

$$\theta(p,r)=\inf(s;\tilde{L}_s^r>p).$$

Then for every stopping time $\theta \leqslant \theta(p, r)$ we deduce

$$E[\sup_{t<\theta} |(X_1^\delta - X)_t^{\sigma-}|^2] \leqslant K(p, r)\left\{h^r(|\delta|) + E\left[\int_{[0,\theta)} (\rho+\rho_1)(\sup_{s<t} |(X_1^\delta - X)_s^{\sigma-}|^2)\, d\tilde{L}_s^r\right]\right\}$$

so by Lemma 3.1(i_1) we have

$$\lim_{|\delta|\mapsto 0} E[\sup_{s<\theta(p,r)} |(X_1^\delta - X)_s^{\sigma-}|^2] = 0 \qquad \text{for all } p, k, r. \tag{3.33}$$

Utilising (3.31)–(3.33) we arrive at

$$\begin{aligned}
&P(\sup_{s\leqslant T} |X_1^\delta(s) - X(s)| \geqslant \varepsilon) \\
&\quad \leqslant P(\sup_{s<\theta(p,r)\wedge\alpha(k)} |X_1^\delta(s) - X(s)| \geqslant \varepsilon) + P(\theta(p,r)\wedge\alpha(k) < T) \\
&\quad \leqslant P(\sup_{s<\theta(p,r)} |(X_1^\delta - X)_s^{\sigma-}| \geqslant \varepsilon) \\
&\qquad + P(\theta(p,r)\wedge\alpha(k) < T) + P(\sigma < \alpha(k)\wedge\theta(p,r)) \\
&\quad \leqslant P(\theta(p,r)\wedge\alpha(k) < T) + P(\sup_{s<\theta(p,r)\wedge\alpha(k)} |X(s)| \geqslant r) \\
&\qquad + P(\sup_{s<\theta(p,r)\wedge\alpha(k)} |X_1^\delta(s)| \geqslant r) + P(\sup_{s<\theta(p,r)} |(X_1^\delta - X)_s^{\sigma-}| \geqslant \varepsilon) \\
&\quad \leqslant P(\theta(p,r)\wedge\alpha(k) < T) \\
&\qquad + P(\sup_{s<\theta(p,r)} |(X_1^\delta - X)_s^{\sigma-}| \geqslant \varepsilon) + 2C(k, T)/r^2.
\end{aligned}$$

Now letting $|\delta| \mapsto 0$, $p, r, k \mapsto \infty$ in this inequality we get

$$\overline{\lim_{|\delta|\mapsto 0}} P(\sup_{s\leqslant T} |X_1^\delta(s) - X(s)| \geqslant \varepsilon) = 0.$$

(b) Let $0<\varepsilon<1$, $r>0$, and $\tilde{L}^r$, $\alpha(k)$, $\theta(p, r)$ be defined as in (a). Define the stopping times

$$\sigma(\varepsilon, \delta) = \inf(t; |(X_3^\delta - X)(t)| \geqslant \varepsilon), \ \sigma_1(r) = \inf(t; |X(t)| \geqslant r-1),$$

$$\sigma = \sigma(\varepsilon, \delta) \wedge \sigma_1(r) \wedge \alpha(k) \wedge \theta(p, r).$$

By the domination property we have for any stopping time τ

$$\begin{aligned}
&E[\sup_{t\leqslant\tau} |(X_3^\delta - X)_t|^2] \\
&\quad \leqslant 2C(p, r)\left\{E[\sup_{t\leqslant\sigma} |X_3^\delta(t) - X_2^\delta(t)|^2] \right.\\
&\qquad \left. + E\left[\int_0^\tau (\rho+\rho_1)(\sup_{s\leqslant t} |(X_3^\delta - X)_s^\sigma|^2)\, d\tilde{L}_t^r\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&\varphi(\delta, p, r, k)\\
&\quad = \sup_{t \leqslant \sigma} |X_3^\delta(t) - X_2^\delta(t)|^2\\
&\quad \leqslant 2 \sup_{\substack{t \leqslant \sigma \\ t_i \leqslant t \leqslant t_{i+1}}} \left[\left| \int_{t_i}^{t} F(t, s, X_2^\delta)\, dZ(s) \right|^2 \right.\\
&\qquad \left. + 2 \left| \int_{t_i}^{t} \left(\int_{t_i}^{s} \partial F(s, u, X_2^\delta)/\partial s\, dZ(s) \right) du \right|^2 \right]\\
&\quad = 2\varphi_1(\delta, p, r, k) + 2\varphi_2(\delta, p, r, k).
\end{aligned} \tag{3.34}
$$

Again applying the domination property we obtain

$$
\begin{aligned}
&E[\varphi_1(\delta, p, r, k)]\\
&\quad \leqslant \left\{ \sum_k E\left[\sup_{\sigma \wedge t_i \leqslant t \leqslant \sigma \wedge t_{i+1}} \left| \int_{t_i}^{t} F(t, s, X_2^\delta)\, dZ(s) \right|^4 \right] \right\}^{1/2}\\
&\quad \leqslant K(p, r) \{ E[\max_k (\tilde{L}^r_{\sigma \wedge t_{i+1}} - \tilde{L}^r_{\sigma \wedge t_i})] \}^{1/2} \mapsto 0 \text{ as } |\delta| \mapsto 0 \qquad \text{for all } p, r,
\end{aligned}
$$

by the continuity of $\tilde{L}^r$ and the dominated convergence theorem. Similarly we obtain

$$\lim_{|\delta| \to 0} E[\varphi_2(\delta, p, r, k)] = 0.$$

The above computation shows that (3.34) becomes

$$
\begin{aligned}
&E[\sup_{t \leqslant \tau} |(X_3^\delta - X)_t^\sigma|^2]\\
&\quad \leqslant \psi(\delta, p, r, k) + C_1(p, r) E\left[\int_0^\tau (\rho + \rho_1)(\sup_{s \leqslant t} |(X_3^\delta - X)_s^\sigma|^2)\, d\tilde{L}_t^r \right],
\end{aligned} \tag{3.35}
$$

where $\lim_{|\delta| \mapsto 0} \psi(\delta, p, r, k) = 0$. By Lemma 3.1($\mathrm{i}_1$) we deduce

$$\lim_{|\delta| \to 0} E[\sup_{t \leqslant \theta(p,r)} |(X_3^\delta - X)_t^\sigma|^2] = 0 \qquad \text{for all } p, k, r.$$

Next the reasoning continues as in (a).

Remark 4.2. Theorem 4.1 covers some results of Gihman and Skorohod [9, Theorems 1 and 2, p. 429–438].

REFERENCES

1. M. BERGER AND . V. MIZEL, Volterra equations with Ito integrals, *J. Integral Equations* **2** (1980), 187–245.
2. M. BERGER AND L. V. MIZEL, Volterra equations with Ito integrals, II, *J. Integral Equations* **2** (1980), 319–337.
3. K. BICHTELER, Stochastic integration and L^p-theory of semimartingales, *Ann. Probab.* **9** (1981), 49–89.
4. G. DA PRATO, M. IANNELLI, AND L. TUBARO, Dissipative functions and finite dimensional stochastic differential equations, *J. Math. Pures Appl.* **57** (1978), 173–180.
5. C. DELLACHERIE AND P. A. MEYER, "Probabilities and Potential, B: Theory of Martingales," North-Holland, Amsterdam, 1980.
6. C. DOLEANS-DADE AND P. A. MEYER, Equations différentielles stochastiques, *Sém. Probab. XI, Lecture Notes in Math.* **581** (1977).
7. M. EMERY, Stabilité des solutions des équations différentielles stochastiques: Application aux intégrales multiplicatives stochastiques, *Z. Wahrsch. Verw. Gebiete* **41** (1978), 241–262.
8. M. EMERY, Equations différentielles stochastiques lipschitziennes; Étude de la stabilité, *Sém. Probab. XIII, Lecture Notes in Math.* **721** (1979).
9. I. V. GIHMAN AND A. V. SKOROHOD, "Stochastic Differential Equations and Applications," Naukova Dumka, Kiev, 1982. [in Russian]
10. I. GYONGY AND N. V. KRYLOV, On stochastic equations with respect to semimartingales I, *Stochastics* **4** (1980), 1–21.
11. I. ITO, On the existence and uniqueness of solutions of stochastic integral equations of the Volterra type, *Kodai Math. J.* **2** (1979), 158–170.
12. J. JACOD AND J. MEMIN, Weak and strong solutions of stochastic differntial equations: Existence and stability, *Lecture Notes in Math.* **851** (1981).
13. M. L. KLEPTSYNA AND A. JU. VERETENNIKOV, On the strong solutions of Ito–Volterra stochastic equations, *Theory Probab. Appl.* **29** (1984), 154–158. [in Russian]
14. T. N. KRAVETS, On the existence and uniqueness of solutions of stochastic integral equations of Volterra type, *heory of Random Processes* **8** (1980), 83–91. [in Russian]
15. T. N. KRAVETS, Weak solutions of stochatic integral equations of Volterra type, *Theory of Random Processes* **9** (1981), 55–60. [in Russian]
16. A. M. KOLODII, On the existence of solutions of stochastic Volterra integral equations, *Theory of Random Processes* **11** (1983), 51–57. [in Russian]
17. E. J. MCSHANE, Stochastic differential equations, *J. Multivariate Anal.* **5** (1975), 121–177.
18. A. V. MEL'NIKOV, Stochastic equations and Krylov's estimates for semimartingales, *Stochastics* **10** (1983), 81–102.
19. M. METIVIER AND J. PELLAUMAIL, "Stochastic Integration," Academic Press, New York, 1980.
20. J. PELLAUMAIL, Solutions faibles et semimartingales, *Sém. Probab. XV, Lecture Notes in Math.* **850** (1981).
21. M. PRATELLI, Majoration dans L^p du type Metivier–Pellaumail pour les semimartingales, *Sém. Probab. XVII, Lecture Notes in Math.* **986** (1983).
22. P. PROTTER, On the existence, uniqueness, convergence and explosions of solutions of systems of stochastic integral equations, *Ann. Probab.* **5** (1977), 243–261.
23. P. PROTTER, Right-continuous solutions of systems of stochastic integral equations, *J. Multivariate Anal.* **7** (1977), 204–214.
24. P. PROTTER, H^p-stability of solutions of stochastic differential equations, *Z. Wahrsch. Verw. Gebiete* **44** (1978), 337–352.
25. P. PROTTER, Volterra equations driven by semimartingales, *Ann. Probab.* **13** (1985), 519–530.

26. P. PROTTER, A comparison of stochastic integrals, *Ann. Probab.* **7** (1975), 276–289.
27. C. TUDOR, Successive approximations for solutions of stochastic integral equations of Volterra type, *J. Math. Anal. Appl.* **103** (1984), 27–37.
28. C. TUDOR, Sur les solutions fortes des équations différentielles stochastiques, *C. R. Acad. Sci. Paris*, **299**, no. 4 (1984), 117–120.
29. R. Q. WY AND X. MAO, Existence and uniqueness of the solutions of stochastic differential equations, *Stochastics* **11** (1983), 19–32.